Mass function and particle creation in Schwarzschild-de Sitter spacetime

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Abstract

We construct a mass function for the Schwarzschild-de Sitter black hole spacetime everywhere in between the black hole and the cosmological event horizon. The mass function is positive definite, continuous and monotonically increasing with the radial distance from the black hole horizon and can be related to the geodesic motion. Finally, choosing a suitable timelike hypersurface in between the two horizons, we study particle creation for a massless quantum scalar field in this spacetime.

Keywords: de Sitter, positive mass, Smarr formula, particle creation

1 Introduction

In general relativity, an important concept is that of a mass function. The principal features of any mass function should be threefold. First, it must be defined with respect to a timelike translational Killing vector field; second, one should be able to relate the mass to the geodesic motion for a Newtonian interpretation and finally, the mass function must be a positive definite quantity. It is the third criterion that makes it a difficult problem to define a mass function because one cannot define a satisfactory notion of conserved gravitational energy-momentum tensor unless one goes to the asymptotic region (see e.g. [1] and references therein). Only an approximate notion of gravitational Hamiltonian density can be defined perturbatively and locally but the positivity of this quantity is far from obvious.

For asymptotically flat spacetimes a gravitational mass can be defined in several ways. The simplest is the Komar mass. This is proportional to the surface integral of the derivative of the

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norm of the timelike Killing field and thus is related directly to geodesic motion. In general the Komar integral will be positive definite only if the matter energy-momentum tensor T_{ab} satisfies the strong energy condition (SEC) : $\left(T_{ab} - \frac{1}{2}T g_{ab}\right)\xi^a \xi^b \geq 0$, for any timelike ξ^a [1].

The second is the Arnowitt-Deser-Misner (ADM) [2, 3, 4] formalism. In this approach a gravitational Hamiltonian density is defined with respect to the background timelike Killing vector field in the asymptotic region and the integral of this Hamiltonian density is computed in the asymptotic region. This integral is interpreted as the gravitational mass.

It is known from the Raychaudhuri equation that geodesics would converge in a mass distribution only if the latter satisfies the SEC [1, 5, 6]. Also, it is known that the SEC implies the weak energy condition, i.e. the positivity of the energy density. Using these two facts a third approach to define gravitational mass and to prove its positivity was developed in [7, 8] for asymptotically flat spacetimes.

The positivity conjecture of the ADM mass was first proved in [9, 10]. Soon afterward, a remarkable proof of the positivity of the ADM mass appeared in [11] using a spacelike spinor field on a spacelike non-singular Cauchy surface. This result was generalized for black holes in asymptotically flat or anti-de Sitter spacetimes in [12]. The $\Lambda \leq 0$ spacetimes usually have well defined asymptotic structure or infinities which are accessible to the geodesic observers. The references mentioned above consider explicit asymptotic structures of such spacetimes at spacelike infinities which are uniquely Minkowskian or anti-de Sitter. In fact the positivity of the ADM mass for $\Lambda \leq 0$ physically reasonable spacetimes admitting spin and well defined asymptotic structures is well understood so far.

But recent observations suggest that there is a strong possibility that our universe is endowed with a small but positive cosmological constant Λ [13, 14]. We note that, since a positive Λ violates SEC, it repels geodesics (see e.g. [15, 16, 17, 18, 19]) and hence the first and the third of the methods mentioned above to define gravitational mass do not seem to be applicable in this case. Also, the known exact stationary solutions with $\Lambda > 0$ (see e.g. [20]) usually exhibits an outer horizon, namely the cosmological event horizon. The tiny observed value of Λ (of the order of $10^{-52}\mathrm{m}^{-2}$) sets the length scale of the horizon to be $\mathcal{O}\left(\frac{1}{\sqrt{\Lambda}}\right)$, which is of course very large but finite. If a black hole is present, it will be located inside the cosmological horizon and the spacetime is known as de Sitter black hole spacetime. The cosmological event horizon acts in these spacetimes as an outer causal boundary [21], beyond which the timelike Killing vector field becomes spacelike and communication is not possible along a future directed causal path thereby ruling out any precise notion of asymptotics.

The very first approach to define mass in de Sitter black hole spacetimes appeared in [21], where a mass function was defined on the black hole and cosmological horizons using the integral of their respective surface gravities. The variation of this mass function gave a Smarr formula. In [22], metric perturbation was considered in a region far away from the black hole but inside the cosmological event horizon where the background spacetime is de Sitter. A local gravitational energy momentum tensor was constructed and with respect to the background de Sitter timelike Killing field the mass of the perturbation was defined. This perturbative approach has much similarity with the usual Hamiltonian formulation of general relativity. We shall adopt this approach explicitly in this paper. For asymptotically Schwarzschild-de Sitter spacetimes the mass in this asymptotic region with respect to the de Sitter background was found to be M, i.e. the mass parameter of the Schwarzschild-de Sitter metric. The spinorial proof of positivity of ADM mass for asymptotically flat spacetimes [11] was generalized later in [23, 24] to show that the mass thus defined in the sense of [22] with respect to the background de Sitter spacetime is indeed a positive definite quantity.

What do we wish to do then? Since there exists no preferred asymptotic region for defining mass we shall define this everywhere in the region between the black hole and the cosmological horizon, keeping in mind that firstly it must be a continuous positive definite quantity. Secondly, since positive Λ corresponds to a positive energy density, the Λ part of this mass function should

increase monotonically with radial distance from the black hole horizon. Thirdly this mass function must be related to the geodesic motion in order to have a satisfactory Newtonian interpretation.

We outline the basic scheme now. We shall set $c = k_{\rm B} = G = \hbar = 1$ throughout. The metric signature will be taken mostly positive $(-,\ +,\ +,\ +)$. If not otherwise mentioned, repeated indices will be summed over. Let us consider the Schwarzschild-de Sitter spacetime written in spherical polar coordinates,

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1}$$

For $3M\sqrt{\Lambda} < 1$, this spacetime has three Killing horizons,

$$r_{\rm H} = \frac{2}{\sqrt{\Lambda}} \cos\left[\frac{1}{3}\cos^{-1}\left(3M\sqrt{\Lambda}\right) + \frac{\pi}{3}\right],\ r_{\rm C} = \frac{2}{\sqrt{\Lambda}} \cos\left[\frac{1}{3}\cos^{-1}\left(3M\sqrt{\Lambda}\right) - \frac{\pi}{3}\right],\ r_{\rm U} = -\left(r_{\rm H} + r_{\rm C}\right). \tag{2}$$

$r_{\rm H}$ is the black hole event horizon and $r_{\rm C} > r_{\rm H}$ is the cosmological horizon, whereas $r_{\rm U}$ is unphysical. For $3M\sqrt{\Lambda} \ll 1$, it is easy to see that $r_{\rm H} \to 2M$ and $r_{\rm C} \to \sqrt{\frac{3}{\Lambda}}$.

As mentioned earlier, the gravitational mass of the perturbation over the de Sitter background has been defined and computed earlier in [22] in a region where $\left(1 - \frac{\Lambda r^2}{3}\right) \gg \frac{2M}{r}$ and it turned out to be M. Instead, we divide the region between the black hole and the cosmological event horizon $(r_{\rm H} < r < r_{\rm C})$ into three consecutive regions of perturbation

$$1 \gg \left(\frac{2M}{r} + \frac{\Lambda r^2}{3}\right) \ \ \text{(Region I)}, \quad \left(1 - \frac{\Lambda r^2}{3}\right) \gg \frac{2M}{r} \ \ \text{(Region II)}, \quad \left(1 - \frac{2M}{r}\right) \gg \frac{\Lambda r^2}{3} \ \text{(Region III)}, \tag{3}$$

where in the first inequality each term on the right hand side is much smaller than unity, the term on the right hand side of the second inequality is much smaller than each of the terms on the left hand side and so for the third inequality. In this sense the three regions are distinct. These three regions can respectively be interpreted as perturbations over background Schwarzschild, Minkowski and de Sitter spacetimes. For $\Lambda \sim 10^{-52}{\rm m}^{-2}$ and M ranging between the extremes 10^4m to 10^9m, it can be easily checked that the above regions exist and are merged smoothly in between. We shall compute the gravitational mass for each region following [22]. Then by defining the gravitational mass of the background in a suitable way, we shall construct the total mass function of the spacetime. The perturbation scheme obviously fails near the horizons, $\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \to 0$, and we shall treat these regions separately.

After we define the mass function, we will connect this to timelike geodesic motion. We shall compute its variation to get a Smarr formula. We shall further present a calculation to determine the spectrum of particles created in this spacetime for a massless quantum scalar field in a suitable timelike hypersurface in a region between the two horizons.

The paper is organized as follows. In the next section we shall construct the total mass function and connect this to geodesics. In Section 3, we shall discuss the Smarr formula and the particle creation. Finally we discuss our results.

2 Construction of the mass function

We start by considering the Λ-vacuum Einstein equations

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = 0, \tag{4}$$

where R_{ab} and R are respectively the Ricci tensor and scalar computed from the metric. Let us assume we can find a region in between the black hole event horizon and the cosmological horizon where the metric g_{ab} can be decomposed in a background $g^{(0)}_{ab}$ and a perturbation γ_{ab} over it

$$g_{ab} = g^{(0)}_{ab} + \gamma_{ab}, \tag{5}$$

where $|\gamma_{ab}| \ll |g^{(0)}_{ab}|$. The basic scheme described in [22] can be outlined as follows : define a local 'gravitational energy-momentum tensor' $T^{(\mathrm{G})}_{ab}$ which consists of quadratic and higher order terms of the γ's, whereas the Einstein tensor consists of $g^{(0)}_{ab}$ and terms linear in γ_{ab}. Let $\nabla^{(0)}_a$ denotes the metric compatible covariant derivative on the background $g^{(0)}_{ab}$. Then the 'energy current' $T^{(\mathrm{G})}_{ab}\xi^{(0)a}$ is conserved with respect to the background, i.e. $\nabla^{(0)}_b \left(T^{(\mathrm{G})\,b}{}_a \xi^{(0)a} \right) \approx 0$, where $\xi^{(0)a}$ is the timelike Killing field corresponding to the background

$$\nabla^{(0)}_a \xi^{(0)}_b + \nabla^{(0)}_b \xi^{(0)}_a \approx 0. \tag{6}$$

Then one computes the flux of the energy current over a closed spacelike hypersurface Σ. We shall apply this method to compute the gravitational mass of the perturbations in different regions of Eq. (3).

Let us start by considering region I of Eq. (3), i.e. linear perturbation over the Minkowski spacetime

$$\begin{aligned} g^{(0)}_{tt} &= -1,\ g^{(0)}_{rr} = 1,\ g^{(0)}_{\theta\theta} = r^2,\ g^{(0)}_{\phi\phi} = r^2 \sin^2\theta, \\ \gamma_{tt} &= \left(\frac{2M}{r} + \frac{\Lambda r^2}{3} \right),\ \gamma_{rr} = \left(\frac{2M}{r} + \frac{\Lambda r^2}{3} \right),\ \gamma_{\theta\theta} = 0 = \gamma_{\phi\phi}, \end{aligned} \tag{7}$$

and the components of the background Killing vector field,

$$\xi^{(0)\mu} \equiv \{1,\ 0,\ 0,\ 0\},\quad \xi^{(0)}_\mu \equiv \{-1,\ 0,\ 0,\ 0\}. \tag{8}$$

The Ricci tensor and scalar reads

$$\begin{aligned} R_{ac} &= R^{(0)}_{ac} + \frac{1}{2}\left[\nabla^{(0)e}\left(\nabla^{(0)}_a \gamma_{ce} + \nabla^{(0)}_c \gamma_{ae} - \nabla^{(0)}_e \gamma_{ac} \right) - \nabla^{(0)}_a \left(\nabla^{(0)f}\gamma_{cf} + \nabla^{(0)}_c \gamma - \nabla^{(0)d}\gamma_{cd} \right) \right] \\ &+\mathcal{O}(\gamma^2) + \ldots, \\ R &= R^{(0)} + \left[\nabla^{(0)e}\nabla^{(0)c}\gamma_{ce} - \nabla^{(0)e}\nabla_{(0)e}\gamma \right] + \mathcal{O}(\gamma^2) + \ldots, \end{aligned} \tag{9}$$

where the trace is defined with respect to $g^{(0)}_{ab}$:$\gamma = \gamma_{ab} g^{(0)ab}$. Also, for the Minkowski background which is a $\Lambda = 0$ vacuum, the Einstein equations become identities

$$R^{(0)}_{ab} - \frac{1}{2} R^{(0)} g^{(0)}_{ab} = 0. \tag{10}$$

Now we use Eq.s (9) and (10) to expand Einstein's equations (4) with $T^{\mathrm{M}}_{ab} = 0$. We shift the $\mathcal{O}(\gamma^2)$ and other higher order terms to the right hand side of Eq. (4) which define the gravitational energy-momentum tensor $\left(T^{(\mathrm{G})}_{ab} - \Lambda g_{ab} \right)$

$$\frac{1}{2}\left[\nabla^{(0)d}\nabla^{(0)}_a \overline{\gamma}_{bd} + \nabla^{(0)d}\nabla^{(0)}_b \overline{\gamma}_{ad} - \nabla^{(0)d}\nabla^{(0)}_d \overline{\gamma}_{ab} - \left(\nabla^{(0)d}\nabla^{(0)c}\overline{\gamma}_{cd} \right) g^{(0)}_{ab} \right] = 8\pi T^{(\mathrm{G})}_{ab} - \Lambda g_{ab}, \tag{11}$$

where $\overline{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\gamma g^{(0)}_{ab}$.

The gravitational mass (M_{G}) is defined as the integral of the 'energy current' $\left(T^{\mathrm{G}}_{tb} - \Lambda g_{tb} \right) \xi^b$ over a spacelike hypersurface Σ orthogonal to the timelike Killing vector field,

$$M_{\mathrm{G}} := \int_\Sigma \left(T^{(\mathrm{G})}_{tb} - \Lambda g_{tb} \right) \xi^b d\Sigma, \tag{12}$$

where 't' corresponds to the direction of the timelike Killing field. Following [22], we obtain the following expression for the energy current from Eq. (11) after a little algebra,

$$T_{tb}^{(\mathrm{G})}\xi^b = \frac{1}{16\pi}\left[\nabla^{(0)d}\left(\left(\nabla^{(0)c}H_{tbcd}\right)\xi^b\right) - \nabla^{(0)c}\left(H_{tbcd}\nabla^{(0)d}\xi^b\right)\right], \tag{13}$$

where

$$H_{abcd} = \left(g_{ca}^{(0)}\overline{\gamma}_{bd} - g_{cd}^{(0)}\overline{\gamma}_{ab} - g_{ab}^{(0)}\overline{\gamma}_{cd} + g_{bd}^{(0)}\overline{\gamma}_{ca}\right). \tag{14}$$

H_{abcd} is antisymmetric under the interchange of $(a,\ d)$ and $(b,\ c)$. Then since in Eq. (13) the indices are fixed $a = t = b$, i.e. timelike, the indices $(d,\ c)$ must be spacelike. Therefore we can convert the integral in Eq. (12) into a surface integral

$$M_{\mathrm{G}} = \frac{1}{16\pi}\left[\oint\left(\nabla^{(0)c}H_{tbcd}\right)\xi^b dS^d - \oint H_{tbcd}\left(\nabla^{(0)d}\xi^b\right)dS^c\right], \tag{15}$$

where 'dS' denotes the volume element of a closed 2-surface, i.e. the boundary of Σ of the region of interest Σ. Since Schwarzschild-de Sitter spacetime is spherically symmetric, the closed surface is a 2-sphere. Also, Eq. (7) gives

$$\gamma = \gamma_{ab}g^{(0)ab} = 0. \tag{16}$$

We now explicitly evaluate Eq. (15) using Eq.s (7), (8) and (16). The covariant derivative on the background Killing vector field is $\nabla_{(0)d}\xi_b = \partial_d\xi_b - \Gamma^e_{db}\xi_e$ where we keep in mind that d and b are summed over as in the equation. Since H_{abcd} is antisymmetric under the interchange of a and d, and a is timelike, d must be spacelike. Keeping in mind $b = t$, it is clear that $\nabla_{(0)d}\xi_b$ is non-vanishing only when $d = r$. We also have $c = r$ in the second integral of Eq. (15). But Eq.s (14), (7) and (16) give $H_{ttrr} = 0$. Thus the second integrand in Eq. (15) is identically vanishing. Now expanding the covariant derivative in the first integral we find that the only non-vanishing term is $-g^{ce}\Gamma^f_{ce}\gamma_{fd}$. Since the direction '$d$' is along r, we finally get

$$M_{\mathrm{G}} = M + \frac{\Lambda r^3}{6}\ . \tag{17}$$

Next we consider perturbation over the de Sitter background, i.e. region II : $\left(1 - \frac{\Lambda r^2}{3}\right) \gg \frac{2M}{r}$, in Eq. (3). This was explicitly done in [22]. The perturbation and the background Killing field are the following

$$\begin{aligned} g_{tt}^{(0)} &= -\left(1 - \frac{\Lambda r^2}{3}\right),\ g_{rr}^{(0)} = \left(1 - \frac{\Lambda r^2}{3}\right)^{-1},\ g_{\theta\theta}^{(0)} = r^2,\ g_{\phi\phi}^{(0)} = r^2\sin^2\theta, \\ \gamma_{tt} &= \frac{2M}{r},\ \gamma_{rr} = \frac{2M}{r\left(1 - \frac{\Lambda r^2}{3}\right)^2},\ \gamma_{\theta\theta} = 0 = \gamma_{\phi\phi}, \end{aligned} \tag{18}$$

and

$$\xi^{(0)\mu} \equiv \{1,\ 0,\ 0,\ 0\},\quad \xi_\mu^{(0)} \equiv \left\{-\left(1 - \frac{\Lambda r^2}{3}\right),\ 0,\ 0,\ 0\right\}. \tag{19}$$

The calculation of the mass of the perturbation is essentially the same as before. The only difference we have to remember is that the de Sitter background we are considering now is a Λ-vacuum

$$R_{ab}^{(0)} - \frac{1}{2}R^{(0)}g_{ab}^{(0)} + \Lambda g_{ab}^{(0)} = 0. \tag{20}$$

The calculation of the mass of the perturbation, using Eq.s (4), (9) and (20) and following the method described above gives

$$M_{\rm G} = M. \tag{21}$$

Thus the mass of the perturbation with respect to the background de Sitter spacetime is given by the mass parameter of the Schwarzschild-de Sitter spacetime.

Finally we consider region III in Eq. (3), i.e. perturbation of the Schwarzschild background by a Λ term. The background and the perturbation are

$$\begin{aligned} g^{(0)}_{tt} &= -\left(1-\frac{2M}{r}\right),\ g^{(0)}_{rr} = \left(1-\frac{2M}{r}\right)^{-1},\ g^{(0)}_{\theta\theta} = r^2,\ g^{(0)}_{\phi\phi} = r^2\sin^2\theta, \\ \gamma_{tt} &= \frac{\Lambda r^2}{3},\ \gamma_{rr} = \frac{\Lambda r^2}{3\left(1-\frac{2M}{r}\right)^2},\ \gamma_{\theta\theta} = 0 = \gamma_{\phi\phi}, \end{aligned} \tag{22}$$

and

$$\xi^{(0)\mu} \equiv \{1,\ 0,\ 0,\ 0\},\ \xi^{(0)}_{\mu} \equiv \left\{-\left(1-\frac{2M}{r}\right),\ 0,\ 0,\ 0\right\}. \tag{23}$$

We follow the same procedure described after Eq. (16). The second integral in Eq. (15) can be shown to be vanishing as earlier and evaluating the first integral we get

$$M_{\rm G} = \frac{\Lambda r^3}{6}\ . \tag{24}$$

The mass functions of Eq.s (17), (21), (24) are clearly positive definite, but are not continuous. In order to achieve a continuous mass function we introduce the notion of the total mass function in the following way. First we note that for the Minkowski background, the background curvature is identically vanishing. Therefore we take the background mass to be zero. For the Schwarzschild background we define the background mass to be the Komar mass,

$$M_{\rm B} = -\frac{1}{8\pi}\oint \nabla_a \xi^{(0)}_t dS^a, \tag{25}$$

where the integration is performed over a 2-sphere. For Schwarzschild spacetime, $M_{\rm B} = M$ anywhere inside that particular perturbation region.

For the de Sitter background we treat the $-\Lambda g^{(0)}_{ab}$ term appearing in the Einstein equations as the energy-momentum tensor ($8\pi T^{\Lambda}_{ab}$) corresponding to the cosmological constant. Thus the corresponding energy current becomes

$$T^{\Lambda}_{ab}\xi^{(0)b} = -\frac{\Lambda}{8\pi} g^{(0)}_{ab}\xi^{(0)b}. \tag{26}$$

But we have from the unperturbed Λ-vacuum equation (20),

$$T^{\Lambda}_{ab}\xi^{(0)b} = -\frac{1}{8\pi} R^{(0)}_{ab}\xi^{(0)b} = \frac{1}{8\pi}\nabla^{(0)d}\nabla^{(0)}_{d}\xi^{(0)}_{a}, \tag{27}$$

using the Killing identity. Thus the gravitational mass $M_{\rm B}$ corresponding to the background de Sitter vacuum is given by

$$M_{\rm B} = \int T^{\Lambda}_{tb}\xi^{(0)b} d\Sigma = \frac{1}{8\pi}\int \nabla^{(0)d}\nabla^{(0)}_{d}\xi^{(0)}_{t} d\Sigma. \tag{28}$$

Since $\xi^{(0)a}$ is a timelike coordinate Killing field, the index d is spacelike above, as can be seen by antisymmetrizing the covariant derivative on $\xi_t^{(0)}$. So we can convert the integral in Eq. (28) into a surface integral over a 2-sphere to get

$$M_{\rm B}=\frac{1}{8\pi}\oint\nabla_d^{(0)}\xi_t^{(0)}dS^d=\frac{\Lambda r^3}{6}\ . \tag{29}$$

We now combine the gravitational mass of each region with the respective background mass to get a total mass function

$$U(r,M)=M_{\rm B}+M_{\rm G}=M+\frac{\Lambda r^3}{6}\ , \tag{30}$$

which is positive definite and monotonically increases with r, thereby encompassing the satisfaction of weak energy condition by positive Λ.

The perturbation theory fails as one moves close to the horizons

$$\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)=0, \tag{31}$$

since we cannot naturally decompose the spacetime in two parts among which one contains a background timelike Killing vector field. We can bypass this difficulty as the following for $3M\sqrt{\Lambda}\ll 1$, which is of course the physically reasonable situation. Let us first consider the region infinitesimally close to the cosmological event horizon. Instead of taking Schwarzschild-de Sitter spacetime, we assume a solution of the kind

$$ds^2=-\left(1-\frac{2M}{r}+\epsilon-\frac{\Lambda r^2}{3}\right)dt^2+\left(1-\frac{2M}{r}+\epsilon-\frac{\Lambda r^2}{3}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2, \tag{32}$$

where ϵ is a dimensionless infinitesimal positive constant. It can be checked that this metric satisfies Einstein's equation. The effect of ϵ is to make the cosmological event horizon larger and the black hole smaller. Now, near the cosmological horizon the quantity $\frac{2M}{r}$ is itself an infinitesimal. We choose $\epsilon\sim\mathcal{O}\left(\frac{2M}{r_{\rm C}}\right)$. It is then obvious that we can choose

$$\begin{aligned} g_{tt}^{(0)} &= -\left(1-\frac{\Lambda r^2}{3}\right),\ g_{rr}^{(0)}=\left(1-\frac{\Lambda r^2}{3}\right)^{-1},\ g_{\theta\theta}^{(0)}=r^2,\ g_{\phi\phi}^{(0)}=r^2\sin^2\theta, \\ \gamma_{tt} &= \left(\frac{2M}{r}-\epsilon\right),\ \gamma_{rr}=\frac{2M-r\epsilon}{r\left(1-\frac{\Lambda r^2}{3}\right)^2},\ \gamma_{\theta\theta}=0=\gamma_{\phi\phi}, \end{aligned} \tag{33}$$

and the background timelike Killing field $\xi^{(0)a}$ same as described in Eq. (19). With this, we repeat the same procedure described earlier and we take $\epsilon\to 0$ at the end of the calculation to get Eq. (30).

Similar analysis can be performed infinitesimally close to the black hole event horizon (with $\epsilon\sim\mathcal{O}\left(\frac{\Lambda r_{\rm H}^2}{3}\right)$) to get the same mass function. Thus we have constructed a mass function everywhere in the region between the black hole event horizon and the cosmological horizon. The mass function is positive definite, continuous and monotonically increasing with r, i.e. the more we move away, the more positive Λ-energy we get inside.

How is this function $U(r,M)=\left(M+\frac{\Lambda r^3}{6}\right)$ really a mass? More precisely, is it meaningful to call this the mass inside the sphere of radius r? One way to answer this question is by studying timelike geodesics in the Newtonian approximation for (1). For any geodesic in this spacetime, four conserved quantities $E,\ L_1,\ L_2,\ L_3$ can be defined respectively to the timelike Killing field and the three generators of a 2-sphere. The first is the conserved energy and the rest can be regarded as the

components of the orbital angular momentum. Then the timelike geodesic motion can be mapped to an effective 1-dimensional non-relativistic motion of a unit mass test particle with energy $\frac{1}{2}E^2$ as in the Schwarzschild spacetime [1],

$$\frac{1}{2}\dot{r}^2 + \psi(r,\ L) = \frac{1}{2}E^2, \tag{34}$$

where the effective potential $\psi(r,\ L)$ is given by

$$\psi(r,\ L) = \frac{1}{2}\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{L^2}{r^2} + 1\right) = \frac{1}{2} + \frac{L^2}{2r^2} - \frac{U(r,M)L^2}{r^3} - \frac{U(r,M)}{r}, \tag{35}$$

where

$$L^2 = L_1^2 + L_2^2 + L_3^2 = r^4\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right), \tag{36}$$

and $U(r,M)$ is the mass function given by Eq. (30). For a Newtonian interpretation we note that for sufficiently large r, we have

$$\psi(r,\ L) \approx \frac{1}{2} - \frac{\Lambda L^2}{3} - \frac{U(r,M)}{r} + \frac{L^2}{2r^2}. \tag{37}$$

The above equation clearly indicates that a geodesic motion far away from the black hole in the Schwarzschild-de Sitter background can be thought of as a motion under a positive definite, position dependent mass function $U(r,M)$, along with the usual repulsive centrifugal barrier term and some constants.

Alternatively, we can write down the geodesic equation explicitly

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{38}$$

where τ is the timelike parameter along the geodesic, and take its non-relativistic limit. We set $\mu = r$ and consider region I of Eq. (3) and take the limit $\tau \to t$. The equation becomes

$$\frac{d^2r}{dt^2} + \left(\frac{M}{r^2} - \frac{\Lambda r}{3}\right) - \frac{L^2}{r^3} = \frac{d^2r}{dt^2} + \partial_r\left(-\frac{U(r,M)}{r}\right) - \frac{L^2}{r^3} \approx 0, \tag{39}$$

using Eq. (36). The above equation clearly shows again the justification of calling $U(r,M)$ a mass function.

3 Smarr formula and particle creation

We shall now compute the variation of this local and total mass function (30) subject to the change of the black hole mass parameter M and keeping Λ fixed. For $\Lambda = 0$ this gives rise to the idea of black hole thermodynamics [25, 26, 27, 28, 29] (see also [30] for a vast review).

The area of the black hole horizon ($r_{\rm H}$) is given by

$$A_{\rm H} = 4\pi r_{\rm H}^2, \tag{40}$$

which, using Eq. (2) we rewrite as

$$M(A_{\rm H}) = -\frac{4\Lambda}{3}\left(\frac{A_{\rm H}}{16\pi}\right)^{\frac{3}{2}} + \left(\frac{A_{\rm H}}{16\pi}\right)^{\frac{1}{2}}. \tag{41}$$

Now we write the mass function $U(r,M)$ in terms of two new variables: the black hole horizon area $A_{\rm H}$ and volume $V=\frac{4}{3}\pi r^3$ enclosed by an arbitrary sphere of radius r on which we have defined the mass function,

$$U(A_{\rm H},V)=-\frac{4\Lambda}{3}\left(\frac{A_{\rm H}}{16\pi}\right)^{\frac{3}{2}}+\left(\frac{A_{\rm H}}{16\pi}\right)^{\frac{1}{2}}+\frac{\Lambda V}{8\pi}\ , \tag{42}$$

the variation of which gives

$$\delta U(A_{\rm H},V)=\left[-\frac{2\Lambda}{(16\pi)^{\frac{3}{2}}}\left(A_{\rm H}\right)^{\frac{1}{2}}+\frac{1}{2\left(16\pi\right)^{\frac{1}{2}}\left(A_{\rm H}\right)^{\frac{1}{2}}}\right]\delta A_{\rm H}+\frac{\Lambda}{8\pi}\delta V. \tag{43}$$

The surface gravity $\kappa_{\rm H}$ of the black hole horizon is given by

$$\kappa_{\rm H}=\left(\frac{M}{{r_{\rm H}}^2}-\frac{\Lambda r_{\rm H}}{3}\right), \tag{44}$$

combining which with Eq.s (40), (43) gives

$$\delta U(A_{\rm H},V)=\frac{\kappa_{\rm H}}{8\pi}\delta A_{\rm H}+\frac{\Lambda}{8\pi}\delta V. \tag{45}$$

We can do a similar calculation with the cosmological horizon in Eq. (40) to write

$$\delta U(A_{\rm C},V)=-\frac{\kappa_{\rm C}}{8\pi}\delta A_{\rm C}+\frac{\Lambda}{8\pi}\delta V, \tag{46}$$

where $A_{\rm C}$ and $\kappa_{\rm C}$ are the cosmological horizon's area and surface gravity respectively. The term $\frac{\Lambda}{8\pi}$ appearing in either of Eq.s (45), (46) can be interpreted as a negative isotropic pressure due to Λ,

$$T_{ab}=\frac{\Lambda}{8\pi}u_a u_b-\frac{\Lambda}{8\pi}\left(u_a u_b+g_{ab}\right), \tag{47}$$

where u_a is a unit timelike vector along the worldline of fluid particles and the energy density ρ and the isotropic pressure P of the fluid have been identified with $\frac{\Lambda}{8\pi}$ and $-\frac{\Lambda}{8\pi}$ respectively in an orthonormal frame. Combining Eq.s (45) and (46) for the same volume V we get a Smarr formula involving horizon parameters only

$$\kappa_{\rm H}\delta A_{\rm H}+\kappa_{\rm C}\delta A_{\rm C}=0, \tag{48}$$

which was derived earlier in [21, 31], which we have rederived using our local mass function. Note that since both $\kappa_{\rm H}$ and $\kappa_{\rm C}$ are positive, the Smarr formula shows that when the area of the black horizon increases, the area of the cosmological horizon decreases and vice versa, an immediate consequence of the expressions of the horizon lengths, Eq.s (2), which state that for $\delta M>0$, $r_{\rm H}$ increases and $r_{\rm C}$ decreases and vice versa. It is clear then unlike asymptotically flat or anti-de Sitter spacetimes, one cannot interpret $\frac{A_{\rm H}}{4}$ as the entropy of the spacetime. Instead one defines a 'total' entropy $S=\frac{A_{\rm H}+A_{\rm C}}{4}$ [31]. For further discussions on this we refer our readers to [32, 33, 34] which consider the variation of Λ as well.

Since $\kappa_{\rm H}\neq\kappa_{\rm C}$ for $r_{\rm H}\neq r_{\rm C}$, Eq.s (45), (46) show that unlike asymptotically flat or anti-de Sitter spacetimes, there can be no unique thermodynamic interpretation in the Schwarzschild-de Sitter spacetime unless one separates the two horizons using a thermally opaque membrane. In this situation one expects two different thermal equilibrium states of temperatures $\frac{\kappa_{\rm H}}{2\pi}$, $\frac{\kappa_{\rm C}}{2\pi}$ corresponding to the black hole and the cosmological horizons respectively. On the other hand, the variation of the 'total' entropy combined with the variation of the total mass function U gives

$$\delta U=\frac{\kappa_{\rm H}\kappa_{\rm C}}{2\pi\left(\kappa_{\rm H}+\kappa_{\rm C}\right)}\delta\left(\frac{A_{\rm H}+A_{\rm C}}{4}\right)+\frac{\Lambda}{8\pi}\delta V, \tag{49}$$

derived earlier in [32] using the global mass function defined on the horizons. Thus with respect to the total entropy one might expect an 'effective equilibrium' temperature $T = \frac{\kappa_{\rm H}\kappa_{\rm C}}{2\pi(\kappa_{\rm H}+\kappa_{\rm C})}$.

In the following we shall derive the different thermal equilibrium states using Kruskal patches for an eternal Schwarzschild-de Sitter spacetime corresponding to each of the horizons (i.e. corresponding to Eq.s (45), (46)).

It was shown in [35] using canonical quantization and Bogoliubov transformations that a stellar object undergoing gravitational collapse in an asymptotically flat spacetime to form a black hole creates Planckian distribution particles at late times. For massless quantum fields this distribution can be measured at the future null infinity, and found to have a temperature $\frac{\kappa_{\rm H}}{2\pi}$, where $\kappa_{\rm H}$ is the surface gravity of the black hole future horizon. Later this result was rederived using path integral quantization [36]. This most remarkable result, known as Hawking radiation, was further justified by the renormalization of the quantum energy-momentum tensors (see [37, 38] and references therein). We further refer our reader to [39, 40] for excellent reviews on this subject.

For an eternal horizon, there is no scenario for gravitational collapse and there exists a past horizon, in addition to the future horizon. It was shown in [21] using path integrals that for an eternal Schwarzschild-de Sitter spacetime, the black hole and the cosmological horizon create thermal particles with temperatures $\frac{\kappa_{\rm H}}{2\pi}$ and $\frac{\kappa_{\rm C}}{2\pi}$ respectively. In [39], particle creation on both the horizons was studied, showing that there can be non-thermal spectra. In [41] particle creation by a Schwarzschild black hole sitting within a Friedmann-Robertson-Walker (FRW) universe was studied. The FRW universe could be a global de Sitter space itself. See also [42] for study of particle creation in Schwarzschild-de Sitter spacetime via complex path analysis.

We shall derive in the following the particle spectra within both the horizons, $r_{\rm H} \leq r \leq r_{\rm C}$, using canonical quantization. We consider a massless minimally coupled scalar field ψ moving in the Schwarzschild-de Sitter spacetime, and ignore any backreaction. Employing the usual separation of variables, $\psi(t,r,\theta,\phi) = e^{-i\omega t}\frac{f_{\omega lm}(r)}{r}Y_{lm}(\theta,\phi)$, the equation of motion for a single mode becomes

$$-\frac{\partial^2 f_{lm}(r,t)}{\partial t^2} + \frac{\partial^2 f_{lm}(r,t)}{\partial r_*^2} - \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{l(l+1)}{r^2} + \frac{M}{r^3} - \frac{\Lambda}{3}\right) f_{lm}(r,t) = 0, \qquad (50)$$

where we have abbreviated $f(r,t) = e^{-i\omega t} f(r)$ and $r_\star$ is the tortoise coordinate defined by,

$$r_\star = \int \frac{dr}{\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)} = \frac{1}{2\kappa_{\rm H}} \ln\left|\frac{r}{r_{\rm H}} - 1\right| - \frac{1}{2\kappa_{\rm C}} \ln\left|\frac{r}{r_{\rm C}} - 1\right| + \frac{1}{2\kappa_{\rm U}} \ln\left|\frac{r}{r_{\rm U}} + 1\right|, \qquad (51)$$

where $\kappa_{\rm U} = \partial_r\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\Big|_{r=r_{\rm U}}$. Thus $r_\star \to \mp\infty$ as $r \to r_{\rm H},\ r_{\rm C}$ respectively. Since we are ignoring backreaction, we have $l \ll M$. Clearly, to find and work with the exact mode solutions of Eq. (50) seems a formidable task and hence we shall have to choose a suitable region to work in. But before we go into that, we digress briefly to construct suitable coordinate systems for the Schwarzschild-de Sitter spacetime. There are two coordinate singularities located at $r_{\rm H}$ and $r_{\rm C}$, therefore we require two Kruskal-like patches to remove them. We define the usual outgoing and incoming null coordinates $(u,\ v)$ as

$$u = t - r_\star,\ \ v = t + r_\star. \qquad (52)$$

By writing the metric (1) in terms of u and r, it is easy to find that $u \to \pm\infty$ as $r \to r_{\rm H},\ r_{\rm C}$ respectively along an incoming null geodesic, whereas by writing it in terms of v and r gives $v \to \mp\infty$ as $\to r_{\rm H},\ r_{\rm C}$ respectively along an outgoing null geodesic. In terms of the null coordinates $(u,\ v)$ the metric becomes

$$ds^2 = \frac{2M}{r}\left(\frac{r}{r_{\rm H}} - 1\right)\left(\frac{r}{r_{\rm C}} - 1\right)\left(\frac{r}{r_{\rm U}} + 1\right) du dv + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (53)$$

where r as a function of $(u,\ v)$ is understood and can be found from Eq. (52). We now define the Kruskal null coordinates for the black hole event horizon as

$$\overline{u} = -\frac{1}{\kappa_{\rm H}} e^{-\kappa_{\rm H} u},\ \overline{v} = \frac{1}{\kappa_{\rm H}} e^{\kappa_{\rm H} v}, \qquad (54)$$

so that $\overline{u} \to 0,\ -\infty$ as $r \to r_{\rm H},\ r_{\rm C}$ respectively, and $\overline{v} \to 0,\ +\infty$ as $r \to r_{\rm H},\ r_{\rm C}$ respectively. The Kruskal null coordinates for the cosmological event horizon can be defined as

$$\overline{u}' = \frac{1}{\kappa_{\rm C}} e^{\kappa_{\rm C} u},\ \overline{v}' = -\frac{1}{\kappa_{\rm C}} e^{-\kappa_{\rm C} v}, \tag{55}$$

so that $\overline{u}' \to +\infty,\ 0$ as $r \to r_{\rm H},\ r_{\rm C}$ respectively, and $\overline{v}' \to -\infty,\ 0$ as $r \to r_{\rm H},\ r_{\rm C}$ respectively. To summarize, the ranges of the various null coordinate systems are

$$-\infty < u < \infty,\ \ -\infty < v < \infty,\ \ -\infty < \overline{u} \leq 0,\ 0 \leq \overline{v} < \infty,\ 0 \leq \overline{u}' < \infty,\ \ -\infty < \overline{v}' \leq 0. \tag{56}$$

Clearly, there will be both outgoing and incoming mode solutions for the field equation. Since $(\overline{u},\ \overline{u}')$ and $(\overline{v},\ \overline{v}')$ are respectively functions of $(u,\ v)$ only, we have modes in terms of these null coordinates,

$$\begin{aligned}
\psi_{\rm out} &= a_i u_i + a_i^{\dagger} u_i^{\dagger} = \overline{a}_i \overline{u}_i + \overline{a}_i^{\dagger} \overline{u}_i^{\dagger} = \overline{a}'_i \overline{u}'_i + \overline{a}_i'^{\dagger} \overline{u}_i'^{\dagger} \\
\psi_{\rm in} &= b_i v_i + b_i^{\dagger} v_i^{\dagger} = \overline{b}_i \overline{v}_i + \overline{b}_i^{\dagger} \overline{v}_i^{\dagger} = \overline{b}'_i \overline{v}'_i + \overline{b}_i'^{\dagger} \overline{v}_i'^{\dagger}, \\
\psi &= \psi_{\rm in} + \psi_{\rm out},
\end{aligned} \tag{57}$$

where $(u_i,\ v_i)$, $(\overline{u}_i,\ \overline{v}_i)$ and $(\overline{u}'_i,\ \overline{v}'_i)$ are modes corresponding to the coordinates $(u,\ v)$, $(\overline{u},\ \overline{v})$ and $(\overline{u}',\ \overline{v}')$ respectively. The index 'i' corresponds to all discrete and continuous indices. The complex quantities a_i etc. are expansion coefficients and as in flat spacetime, they are interpreted as creation and annihilation operators associated with respective modes. Note that for an eternal Schwarzschild-de Sitter spacetime, there should be only ingoing modes at $r = r_{\rm H}$ and only outgoing modes at $r = r_{\rm C}$. In that sense the mode expansions in Eq.s (57), which contain both ingoing and outgoing modes, are complete.

The creation and annihilation operators are defined to satisfy the commutation relations

$$\begin{aligned}
&\left[a_i,\ a_j^{\dagger}\right] = \delta_{ij},\ \ [a_i,\ a_j] = 0 = \left[a_i^{\dagger},\ a_j^{\dagger}\right],\ \ \left[b_i,\ b_j^{\dagger}\right] = \delta_{ij},\ \ [b_i,\ b_j] = 0 = \left[b_i^{\dagger},\ b_j^{\dagger}\right], \\
&\left[\overline{a}_i,\ \overline{a}_j^{\dagger}\right] = \delta_{ij},\ \ [\overline{a}_i,\ \overline{a}_j] = 0 = \left[\overline{a}_i^{\dagger},\ \overline{a}_j^{\dagger}\right],\ \ \left[\overline{b}_i,\ \overline{b}_j^{\dagger}\right] = \delta_{ij},\ \ \left[\overline{b}_i,\ \overline{b}_j\right] = 0 = \left[\overline{b}_i^{\dagger},\ \overline{b}_j^{\dagger}\right], \\
&\left[\overline{a}'_i,\ \overline{a}_j'^{\dagger}\right] = \delta_{ij},\ \ [\overline{a}'_i,\ \overline{a}'_j] = 0 = \left[\overline{a}_i'^{\dagger},\ \overline{a}_j'^{\dagger}\right],\ \ \left[\overline{b}'_i,\ \overline{b}_j'^{\dagger}\right] = \delta_{ij},\ \ \left[\overline{b}'_i,\ \overline{b}'_j\right] = 0 = \left[\overline{b}_i'^{\dagger},\ \overline{b}_j'^{\dagger}\right].
\end{aligned} \tag{58}$$

The inner product of the modes $(u_i,\ v_i)$ are defined as

$$\begin{aligned}
&(u_i,\ u_j) = \frac{i}{2}\int_{\Sigma} \left(u_i^{\dagger}(\nabla_a u_j) - u_j(\nabla_a u_i^{\dagger})\right) d\Sigma^a = \delta_{ij},\ (v_i,\ v_j) = \frac{i}{2}\int_{\Sigma} \left(v_i^{\dagger}(\nabla_a v_j) - v_j(\nabla_a v_i^{\dagger})\right) d\Sigma^a = \delta_{ij}, \\
&(u_i,\ u_j^{\dagger}) = 0 = (v_i,\ v_j^{\dagger}),
\end{aligned} \tag{59}$$

where Σ is suitable hypersurface and the direction 'a' is along its normal. In an asymptotically flat spacetime, one chooses Σ to be the past null infinity. But as we discussed earlier, in presence of a de Sitter horizon, infinities are not accessible to an observer located within that horizon. So we have to choose Σ differently here.

In order to do that let us look at Eq. (50). The quantity $\left(\frac{M}{r^3} - \frac{\Lambda}{3}\right)$ can be written as $\frac{1}{2r}\partial_r\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$. Since the norm $\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)$ vanishes at two points $r_{\rm H} \neq r_{\rm C}$, it must reach at least one maximum in between. This is given by $r_{\rm m} = \left(\frac{3M}{\Lambda}\right)^{\frac{1}{3}}$, which ranges between 10^{18}m to 10^{20}m for $r_{\rm H} \sim 10^4$m to 10^{10}m, and for the observed $\Lambda \sim 10^{-52}\text{m}^{-2}$. In any case, $\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \sim 1$ near this maximum and we are therefore in region I of Eq. (3). Also, since by our assumption $l \ll M$, Eq.s (50), (52), (54), (55) show that around this maximum we have usual outgoing and incoming positive frequency

plane wave modes,

$$u(\omega,l,m)\sim\frac{1}{\sqrt{4\pi\omega}}\frac{e^{-i\omega u}}{r}Y_{lm}(\theta,\ \phi),\qquad v(\omega,l,m)\sim\frac{1}{\sqrt{4\pi\omega}}\frac{e^{-i\omega v}}{r}Y_{lm}(\theta,\ \phi),$$
$$\overline{u}(\omega,l,m)\sim\frac{1}{\sqrt{4\pi\omega}}\frac{e^{-i\omega\overline{u}}}{r}Y_{lm}(\theta,\ \phi),\qquad \overline{v}(\omega,l,m)\sim\frac{1}{\sqrt{4\pi\omega}}\frac{e^{-i\omega\overline{v}}}{r}Y_{lm}(\theta,\ \phi),$$
$$\overline{u}'(\omega,l,m)\sim\frac{1}{\sqrt{4\pi\omega}}\frac{e^{-i\omega\overline{u}'}}{r}Y_{lm}(\theta,\ \phi),\qquad \overline{v}'(\omega,l,m)\sim\frac{1}{\sqrt{4\pi\omega}}\frac{e^{-i\omega\overline{v}'}}{r}Y_{lm}(\theta,\ \phi),\tag{60}$$

along with their negative frequency counterparts. Let us now define the Bogoliubov transformation coefficients (see e.g. [38] for details) and consider the outgoing u and $\overline{u}$ modes first,

$$(\overline{u}_i,\ u_j)=\alpha_{ij},\ \left(\overline{u}_i,\ u_j^\dagger\right)=\beta_{ij}.\tag{61}$$

Let us now consider the equality between the first two mode expansions in the first of Eq.s (57). We use Eq.s (59), (61) and the commutation relations (58) to get

$$\alpha_{ik}\alpha^\dagger_{kj}-\beta_{ik}\beta^\dagger_{kj}=\delta_{ij},\ \alpha_{ik}\beta^\dagger_{kj}-\beta^\dagger_{kj}\alpha_{ik}=0.\tag{62}$$

Subject to these relations and the commutations, one can then take the inverse transformations

$$\overline{a}_i=\alpha_{ij}a_j-\beta^\dagger_{ij}a^\dagger_j.\tag{63}$$

If $|0\rangle$ denotes the vacuum associated with the $(u,\ v)$ modes, then the Kruskal observer will 'see' particles in $|0\rangle$ in the i-th mode,

$$\langle 0|\overline{a}^\dagger_i\overline{a}_i|0\rangle=\Sigma_j\left|\beta_{ij}\right|^2\ \text{(no sum on } i).\tag{64}$$

Note that we are doing this in a region where particles and number operator can have well defined meaning. With all the above equipments, our sole task is now thus to compute β_{ij}. We note that on any $r=$ constant hypersurface, $dt=d(t-r_\star(r))=du=e^{\kappa_{\rm H}u}d\overline{u}$, using Eq.s (54). There will be a $\partial_{r_\star}$ coming from the normal direction of the hypersurface volume element. But $\partial_{r_\star}e^{-i\omega u}=i\omega e^{-i\omega u}=-\partial_u e^{-i\omega u}=-e^{-\kappa_{\rm H}u}\left(\partial_{\overline{u}}e^{-i\omega u(\overline{u})}\right)$, which means for these modes $dt\partial_{r_\star}\equiv d\overline{u}\partial_{\overline{u}}$. Putting these in all together it is straightforward to calculate

$$\alpha_{\omega,\omega'}=\frac{ik}{4\pi\sqrt{\omega\omega'}}\int_{(\Sigma,\ r=r_{\rm m})}\left[\left(\omega'-\frac{\omega}{\kappa_{\rm H}\overline{u}}\right)e^{i\omega'\overline{u}}e^{\frac{i\omega}{\kappa_{\rm H}}\ln(-\kappa_{\rm H}\overline{u})}\right]d\overline{u},\tag{65}$$

where all the constants including those arising from summation of the discrete indices and angular integral have been dumped into the constant k. We are yet to choose the limit of the above integration. An observer sitting on that hypersurface will see null waves moving to and fro. Since on or in the vicinity of that surface the solution has plane wave solutions, the observer will see the phases of those waves to be constant. Since t varies on that surface, we exhaust all the possible values of those constant phases. Thus we choose the limit of $\overline{u}$ in Eq. (65) to be $-\infty$ to 0 (Eq. (56)). With this, the integral in Eq. (65) looks exactly the same as in asymptotically flat spacetime and clearly we have been able to computationally map our problem to the asymptotically flat spacetime case [35, 39, 40], and bypassed all the possible difficulties of a de Sitter spacetime regarding non-trivial mode solutions, asymptotics or even redshift effects. Following [35, 39], we now evaluate the integral. Analytically continuing this to the complex plane, and treating $\overline{u}=0$ as a branch cut one obtains

$$\left|\alpha_{\omega,\omega'}\right|^2=e^{\frac{2\pi\omega}{\kappa_{\rm H}}}\left|\beta_{\omega,\omega'}\right|^2.\tag{66}$$

Then from Eq. (62) we get

$$\int d\omega'\delta(\omega-\omega') = -k' \int d\omega' \left(1 - e^{\frac{2\pi\omega}{\kappa_{\rm H}}}\right) |\beta_{\omega,\omega'}|^2 , \tag{67}$$

where k' is some positive constant arising from summation of the discrete indices and angular integrations. We note that for $\kappa_{\rm H} \to 0$, $\beta \to 0$ and there is no particle creation.

Thus the Kruskal observer will 'see' the $|0\rangle$ vacuum is filled with thermal distribution of outgoing particles around $r = r_{\rm m}$,

$$I(\omega) \sim \frac{1}{e^{\frac{2\pi\omega}{\kappa_{\rm H}}} - 1}, \tag{68}$$

with temperature $T_{\rm H} = \frac{\kappa_{\rm H}}{2\pi}$. Note that unlike the collapsing scenario we do not have any greybody effect here, because we are merely comparing two outgoing modes. Similar analysis for incoming modes $(v,\ \overline{v})$ shows that there are incoming thermal particles with the same temperature too, showing that there will be thermal equilibrium. This justifies the thermodynamic interpretation of Eq. (45). The $\Lambda\delta V$ term can be interpreted as the infinitesimal variation of r around $r = r_{\rm m}$.

Similar calculations with incoming $(v,\ \overline{v}')$ modes show that the p rimed Kruskal observers will detect thermal radiation at temperature $T_{\rm C} = \frac{\kappa_{\rm C}}{2\pi}$. This justifies Eq. (46) also as a thermodynamic equation. We also note that if we set $\kappa_{\rm C} = 0$, there will be no particle creation for the cosmological horizon.

For fermionic field the commutations are replaced with anticommutations and Eq.s (62) is modified with a '+' in place of '−'. This will give Fermi-Dirac distribution with the same respective temperatures.

Let us summarize the results now. We have constructed a positive definite, local and continuous total mass function everywhere in between the two horizons of the Schwarzschild-de Sitter black hole spacetime. The Λ part of the mass function increases monotonically with radial distance, which is in correspondence to the positivity of the vacuum energy density. We have also related the mass function to the geodesic motion. We have rederived all the thermodynamic relations by varying this mass function. Finally we have computed particle creation in a suitable region of this spacetime to verify those thermodynamic relations.

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